The thickness of the film entrained by a fiber quickly withdrawn out of a bath of wetting liquid is of interest. For velocities larger than a threshold (usually of the order of 1 m/s), inertia can no longer be neglected and must be incorporated in a generalized form of the classical Landau-Levich-Deryaguin model. It is shown here that the effect of inertia is to make the film thicker, which agrees with recent observations.

I. INTRODUCTION

In a famous paper, Landau and Levich derived an expression for the thickness of the film entrained by a plate slowly pulled out of a bath of viscous wetting liquid.¹ Deryaguin then adapted the theory to the case of a fiber drawn out of a reservoir.² We try here to incorporate inertia in this theory: in most practical cases (in lubrication processes, for example), the coating velocity is high and inertia cannot be ignored. We first recall qualitatively the classical arguments of Landau, Levich, and Deryaguin (LLD). Then we try to build a model to generalize the LLD theory to the case of non-negligible Reynolds numbers.

II. A SUMMARY OF LANDAU–LEVICH–DERYAGUIN ARGUMENTS

Figure 1 pictures what occurs at the exit of a reservoir of liquid from which a fiber (radius b) is pulled out at a constant velocity \( V \). If the liquid is completely wetting, the fiber comes out coated with a layer of liquid (thickness \( e \)). The region between the entrained film and the part of the static meniscus unperturbed by the motion is referred to as the dynamic meniscus. We note \( \lambda \) its length, which depends on \( V \). For a thin fiber, the static meniscus has a nearly zero curvature.³ The dynamic one has a thickness of order \( e \), and thus a curvature of order \( 1/(b + e) \lambda^2 \). At the crossover between these two menisci, there is no flow and the Laplace pressures are balanced, which gives \( \lambda \). For thin films (\( e \ll b \)), it writes

\[
\lambda = \sqrt{eb}. \tag{1}
\]

Inside the (thin) entrained film, there is a Laplace pressure \( \Delta \rho = \gamma/b \) (\( \gamma \) is the surface tension of the liquid). This super-pressure generates a flow inside the dynamic meniscus, toward the static meniscus. For small Reynolds numbers (i.e., small withdrawal velocities), the flow obeys the Poiseuille law:

\[
V = \frac{\varepsilon^2\gamma}{\mu\lambda b}, \tag{2}
\]

where \( \mu \) is the liquid viscosity. Here \( \lambda \) can be eliminated in Eq. (2) by using Eq. (1), which yields

\[
e = \alpha b \, \text{Ca}^{2/3}, \tag{3}
\]

where \( \text{Ca} \) is the capillary number (\( \text{Ca} = \mu V/\gamma \)) and \( \alpha \) a constant, explicitly calculated in Refs. 1 and 2: \( \alpha = 1.34 \). Equation (3) has been checked experimentally by several authors,² using viscous oils slowly entrained by thin fibers.

A necessary condition for Eq. (3) to be obtained is that both the capillary and the Reynolds numbers must be much smaller than unity. For low withdrawal velocities (of order 0.1 mm/s to 1 cm/s) and viscous oils (as in Ref. 4), these conditions are satisfied. For a high-velocity withdrawal from an aqueous bath (\( V \) of order 1 m/s), the capillary number remains small (less than 0.1), but the Reynolds number becomes of order 1–10, because of the low viscosity of water. Therefore the question is to understand how inertia may modify Eq. (3). This problem should mimic industrial conditions. Just after fibers are made, they are pulled at a velocity ranging between 1 and 10 m/s through a dilute aqueous solution in order to lubricate them.

III. SET OF EQUATIONS

We first write the general equations concerning the flow inside the dynamic meniscus. We call \( r \) and \( z \) the radial and axial coordinates, and \( u_r \) and \( u_z \) the components of the velocity along these coordinates. Dimensionless variables and functions \( \eta, \xi, \nu, U, e, \delta, \kappa, \) and \( \pi \) are introduced by

\[
s = r/b + e \eta, \tag{4a}
\]

\[
z = \lambda \xi, \tag{4b}
\]

\[
u_r = e/\lambda \nu_u, \tag{4c}
\]

\[
u_z = \nu_u, \tag{4d}
\]

\[
S(z) = b + eH(\xi), \tag{4e}
\]

\[
e = e/b, \tag{4f}
\]

\[
\delta = e/\lambda, \tag{4g}
\]

\[
\Delta \rho = \gamma e/\lambda^2 \kappa + \mu \nu_u/\lambda \pi, \tag{4h}
\]

where \( S(z) \) is the equation of the dynamic meniscus. Here \( \kappa \) is the dimensionless total curvature of the interface, which writes

\[
\kappa = (1 + \delta \pi H'^2)^{3/2} \left( -H'' + \frac{e}{3} \frac{1 + \delta \pi H'^2}{1 + eH} \right), \tag{5}
\]

where \( H' \) and \( H'' \) are the first and second derivatives of the dynamic meniscus, respectively.
In these equations, radial space variables are normalized by $e$, and longitudinal ones by $\lambda$, as suggested by Fig. 1. The flow inside the dynamic meniscus is described by the Navier-Stokes equation. Introducing the dimensionless variables, and supposing the fiber vertical ($\rho$ is the specific mass of the liquid and $g$ is the gravity acceleration), it reads as

\[ \Delta u - \frac{\delta^2 \kappa}{Ca} \frac{e^2}{a^2 Ca} \frac{\delta^2}{\delta \xi^2} - \text{Re} \frac{Ca^{1/3}}{a^2 Ca (u \delta \xi + u \delta \eta)}, \]

where $a$ is the capillary length: $a = (\gamma/g \rho)^{1/2}$, usually of order one millimeter. The gravity term where it appears must be taken into account as soon as the fiber is vertical, so that a gravitational flow is superposed to the Laplace flow. Note that the Reynolds number has been defined by using as a characteristic length the thickness $e$: $\text{Re} = e V^3 / \mu$. Finally, the operator $\Delta$ is the cylindrical Laplacian, which writes in the $(\eta, \xi)$ coordinates:

\[ \Delta f = \frac{[(1 + \epsilon \eta) f_{\eta}]_{\eta}}{1 + \epsilon \eta} + \delta^2 f_{\xi \xi}, \]

where the subscripts mean a partial derivation. The description of the problem is completed by writing the equation of continuity:

\[ \frac{\partial}{\partial \xi} [(1 + \epsilon \eta) u] + \frac{\partial}{\partial \eta} [(1 + \epsilon \eta) v] = 0, \]

and the boundary conditions, for $\eta = 0$ (at the fiber surface),

\[ u = 1 \quad \text{and} \quad v = 0, \]

and for $\eta = H$ (at the free interface),

\[ u \eta + \delta^2 \left[ v \xi + 4 \frac{H'}{1 - \delta^2 H'^2} \eta \right] = 0, \]

\[ u H' = v. \]

Introducing the natural scaling for the length of the dynamic meniscus [see Eqs. (1) and (2)], we set $\lambda = e Ca^{-1/3}$, which writes $\delta = Ca^{1/3}$ with the dimensionless variables. For small capillary numbers ($Ca < 10^{-1}$), the $\delta^2$ terms and the viscous term in Eq. (4b) can be neglected. Then Eqs. (6a), (7), (8a), (8b), (8c), and (5) reduce to

\[ u \eta - \frac{\epsilon^2}{a^2 Ca} = \text{Re} \frac{Ca^{1/3}}{a^2 Ca} (u \eta + u \eta), \]

\[ u \xi + u \eta = 0, \]

for $\eta = 0, \ u = 1$, and $v = 0,$

for $\eta = H, \ u \eta = 0,$ and $u H' = v,$

\[ \kappa = -H^2 + e \delta^2 - 2. \]

Eqs. (9b) and (9e) being only valid for $\epsilon < 1$.

**IV. PROFILE OF THE DYNAMIC MENISCUS**

By integrating the Navier-Stokes equation (9a) and the continuity equation (9b) along the radial coordinate from $\eta = 0$ to $\eta = H$, a differential equation for the profile of the dynamical meniscus is obtained:

\[ HH'' - u \eta \eta \eta = \text{Re} \frac{Ca^{1/3}}{a^2 Ca} \left( \int_0^H u \ d \eta \right) + \frac{e^2}{a^2 Ca} H, \]

\[ \frac{d}{d \xi} \left( \int_0^H u \ d \eta \right) = 0. \]

These equations still remain difficult to solve since $u$ depends on both $\xi$ and $\eta$. A method for going further has been proposed by Shkadov in Ref. 5. The velocity profile is supposed to remain parabolic, as in the purely viscous case, but the effective pressure gradient that induces the flow is taken to be unknown. Besides, the velocity profile must satisfy the boundary conditions (9c) and (9d), so that it simply writes

\[ u = 1 + A(\xi) \left[ \frac{\eta}{H} \right]^2. \]

The unknown coefficient $A(\xi)$ is related to the profile $H(\xi)$ by the flux conservation. Hence, we obtain

\[ A(\xi) = 3 \left( \frac{1 - H}{H} \right) \frac{e^2}{a^2 Ca H}. \]

The latter equation takes into account the gravity Poiseuille flow that occurs in the region of constant thickness, in the case of a vertical fiber. In most practical cases, the gravity can be neglected (for example, in the case of horizontal withdrawal, drawn in Fig. 1), and Eq. (12) gets simpler (the second term of the right term is zero). Then, by gathering Eqs. (10), (11), and (12), a third order differential equation for the profile $H(\xi)$ is obtained:

\[ H'^2 H'' - 3(1 - H) + \frac{1}{2} \text{Re} Ca^{1/3} (H^2 - 6) H'. \]

For $\text{Re} = 0$, Eq. (13) is just the Landau–Levich equation for the profile. The parameter $\text{Re} Ca^{1/3}$ increases with the velocity $V$, so that the second term must be considered for describing quick withdrawal. This term describes how inertia perturbs the profile of the dynamic meniscus, and thus modifies the entrained thickness. We define the dimensionless parameter $C$ by setting.
Taking usual values for the different parameters shows that \( C \) becomes of order 1 when \( V \) is of order 1 m/s.

V. RESOLUTION

We first recall the calculation of Landau et al. in order to find the dependence of the thickness \( e \) on \( V \). The matching of curvatures between the apex of the static meniscus (of zero curvature for a thin fiber) and the bottom of the dynamic one writes

\[
e/b = H''|_{H \to \infty} = Ca^{2/3}.
\]

The limit of \( H'' \) for \( H \) tending to infinity is obtained by solving numerically Eq. (13). With \( Re=0 \) (negligible inertia), it was shown by LLD to be equal to 1.34, which leads to Eq. (3).

Compared with the LLD solution, the inertial term brings two difficulties. First, the differential equation of the profile now depends on \( e \) and on other parameters. Second, integration of (13) shows that \( H''(\xi) \) slowly diverges toward infinity for \( H \) tending to infinity. So the asymptotic matching used in Eq. (15) to obtain \( e \) must be abandoned. We now write that the matching between the static and the dynamic meniscus takes place at a point \( \xi^* \) (unknown). At \( \xi = \xi^*, H(\xi) \) and its three first derivatives match the static profile.

Equation (9e) immediately shows that the matching with a region of zero curvature (the static meniscus) leads to

\[
e/b = H''(\xi^*).
\]

The third derivative \( H'''(\xi) \) is calculating by taking the derivative of Eq. (5). Writing that the derivative of the curvature of the static meniscus is zero (since its curvature is zero) leads to (for \( e \leq 1 \))

\[
\xi^2 = Ca^{2/3} = \frac{H''''(\xi^*)}{H''(\xi^*)H''''(\xi^*)}.
\]

The following step of the resolution consists of solving numerically the differential equation (13), for different values of the parameter \( C \). For each value of \( C \), a profile \( H(\xi) \) is obtained, and the derivatives \( H'(\xi), H''(\xi), \) and \( H'''(\xi) \) can be calculated. In Fig. 2, the second derivative \( H''(\xi) \) of the profile is drawn as a function of \( H \) for different values of \( C \) (\( C = 0, 1, \) and \( 5 \)). \( C = 0 \) corresponds to the LLD case. Then \( H'' \) tends to 1.34 when \( H \) tends to infinity. Conversely, as soon as \( C \) is nonzero, \( H'' \) logarithmically diverges with \( H \).

\( C \) (\( C = 0.1, 0.5, 1, 2, 3, 5, \) and \( 10 \)), which are indicated close to the lines. \( C = 10 \) was the highest value studied, since for larger values oscillations appeared when integrating the profile, making the matching irrelevant. In the same figure, the LLD equation (3) is also pictured.

When \( C \) is negligible (\( C \leq 0.1 \)), the corresponding behavior is close to be fitted by the LLD equation, which is logical. When \( C \) cannot be neglected compared with 1, it can be seen that the film is thicker than predicted by Eq. (3). The thickening effect is all the more sizable, since \( C \) is large. But these lines do not allow to see directly the \( e(Ca) \) dependence, since \( C \) itself depends on \( Ca \).

VI. NUMERICAL RESULTS

The results of the computation are displayed in Fig. 3. The value of \( e=e/b \) is plotted versus \( Ca \) for several values of \( C \). The decreasing lines simply express the definition of \( C \) (Eq. (14)). As \( C \) depends on \( Ca \), the actual \( e(Ca) \) behavior is obtained by taking the intersects (stressed by black points) of the two lines for a given value of \( C \). It can thus be seen that inertia causes a thickening of the film.
At a velocity larger than a threshold value, films are indeed thicker than expected, and the deviation increases with the viscosity, and density of the liquid. We have shown that the effect of inertia is to increase the film thickness. The threshold in velocity above which the film thickness increases as the capillary number to the power 2, is found to be of order 5, and to remain nearly constant as a function of C. The thickness at the matching point increases in the same way as the film thickness.

VII. CONCLUDING REMARKS

We have presented a calculation that takes into account inertia in the classical Landau–Levich–Deryaguin derivation of the film thickness entrained by a fiber withdrawn out of a liquid. We have shown that the effect of inertia is to increase the film thickness. The threshold in velocity above which the thickening effect is important is given by C of order 1, where C is a dimensionless quantity, depending on the parameters of the problem (fiber radius and velocity, and surface tension, viscosity, and density of the liquid).

A recent experiment actually indicates that strong deviations from the LLD equation occur when pulling a wire out of pure water (or out of a silicone oil of low viscosity). At a velocity larger than a threshold value, films are indeed thicker than expected, and the deviation increases with the velocity (diverging behavior). By using different liquids (pure water, water containing surfactants, tetradecane, and silicone oil) and various wire radii, the threshold was experimentally checked to be controlled by the parameter C. Nevertheless, a comparison between theory and experiments shows that the calculations underestimate the experimental data, since the latter finally show a diverging behavior rather than a simple scaling law. This disagreement concerns capillary numbers far above the threshold, and is probably due to the fact that the theory only takes into account first-order terms for \( \delta \) (\( \delta \) terms are neglected from Eq. [9]).

If the fiber is pulled at larger velocities, the experiment shows that the diverging behavior first saturates. Then, for still larger velocities, the film thickness slowly decreases as a function of \( V \).

The saturation is probably due to some geometrical constraints: the thickness of the dynamic meniscus diverges with the film thickness, but this divergence can be experimentally limited by the size of the reservoir. In Ref. 6, for example, the reservoir consisted in a tube, of radius \( R = 2 \) mm. In the divergence regime, we could directly observe what happened at the exit of the tube: above a certain velocity (which indeed defined the beginning of the saturation regime), the dynamic meniscus was trapped on the edge of the tube, so that the thickness saturated at a value proportional to \( R \).

In the decreasing regime, the film thickness was shown to vary as \( V^{-1/2} \), which indicates a boundary layer regime. As the fiber enters the reservoir droplet, a viscous boundary layer develops around it. Its thickness increases as \( (\nu t)^{1/2} \), where \( \nu \) is the kinematic viscosity of the liquid, and \( t \) is the time spent in the drop. If the length of the reservoir is called \( L \), we simply have at the exit of the reservoir: \( t = L/V \). Hence, in the high-velocity regime, the thickness of the boundary layer may become smaller than the thickness calculated above, so that the fiber only entrains the boundary layer, which explains the \( V^{-1/2} \) behavior. Conversely, a well-developed boundary layer around the fiber may justify that the velocity profile remains parabolic in the dynamic meniscus, as supposed in Eq. (11).

Note finally that data were also obtained in plate pulling experiments (the withdrawal of a vertical plate out of a bath of liquid). Divergence of the film thickness was not observed, probably because these experiments were done using viscous nills. Then large capillary numbers can be reached even if the withdrawal velocities remain small. Actually, a diverging behavior due to inertia should be also observed in plate withdrawal, even if the effect should be partially damped by gravity.

ACKNOWLEDGMENTS

We thank P. G. de Gennes and I. M. di Meglio for very useful discussions and D. Parseghian for his comments.